Exercise 23

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

Solution

Multiply the numerator and denominator by the reciprocal of the highest power of x in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} &= \lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^3}\sqrt{1+4x^6}}{\frac{1}{x^3}(2-x^3)} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}(1+4x^6)}}{\frac{2}{x^3}-1} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} \\
&= \frac{\lim\limits_{x\to\infty} \sqrt{\frac{1}{x^6}+4}}{\lim\limits_{x\to\infty}\left(\frac{2}{x^3}-1\right)} \\
&= \frac{\sqrt{\lim\limits_{x\to\infty}\left(\frac{1}{x^6}+4\right)}}{\lim\limits_{x\to\infty}\frac{2}{x^3} - \lim\limits_{x\to\infty} 1} \\
&= \frac{\sqrt{\lim\limits_{x\to\infty}\frac{1}{x^6} + \lim\limits_{x\to\infty} 4}}{\lim\limits_{x\to\infty}\frac{2}{x^3} - \lim\limits_{x\to\infty} 1} \\
&= \frac{\sqrt{0+4}}{0-1} \\
&= -2
\end{aligned}$$